Appendix A

Markov Processes and PH Distributions

We recall here the definitions of both discrete time and continuous time Markov processes, as well as some related properties which are often used throughout the text. We also define PH random variables. The interested reader may find material on this topics in Latouche and Ramaswami [29], Norris [38] or Resnick [41], among others.

A.1 Markov Processes

Consider a discrete time stochastic process $\{X_n : n \in \mathbb{N}\}$; it is a family of random variables, indexed by $\mathbb{N}$ and such that $X_n \in \mathcal{S}$ for all $n \in \mathbb{N}$. The set $\mathcal{S}$ is called the state space of the process and is assumed to be denumerable.

Definition A.1.1 *The stochastic process $\{X_n : n \in \mathbb{N}\}$ is a Markov chain if and only if it satisfies the Markov property*

$$P[X_{n+1} = j | X_0, X_1, \ldots, X_n] = P[X_{n+1} = j | X_n]$$

for all $n \in \mathbb{N}$ and $j \in \mathcal{S}$.

The Markov chain is homogeneous if

$$P[X_{n+1} = j | X_n = i] = P[X_1 = j | X_0 = i]$$

for all $n \in \mathbb{N}$ and $i, j \in \mathcal{S}$.

In the continuous time case, we consider a collection of random variables $\{X(t) : t \in \mathbb{R}^+\}$ where for all $t \in \mathbb{R}^+$, $X(t) \in \mathcal{S}$. The state space $\mathcal{S}$ is assumed to be denumerable.

Definition A.1.2 *The stochastic process $\{X(t) : t \in \mathbb{R}^+\}$ is a Markov process if and only if it satisfies the Markov property*

$$P[X(t+s) = j|X(u), 0 \le u \le t] = P[X(t+s) = j|X(t)]$$

for all $s, t \in \mathbb{R}^+$ and $j \in \mathcal{S}$. It is homogeneous if, in addition,

$$P[X(t+s) = j|X(s) = i] = P[X(t) = j|X(0) = i]$$

for all $s, t \in \mathbb{R}^+$ and $i, j \in \mathcal{S}$.

Throughout this work, we essentially deal with continuous time Markov processes, and they are all homogeneous. This is the reason why we only consider this case from now on.

Let $\{X(t) : t \in \mathbb{R}^+\}$ be a homogeneous Markov process. The vector $\boldsymbol{\alpha} = (\alpha_i : i \in \mathcal{S})$ such that $\alpha_i = \mathrm{P}[X(0) = i]$ for i in $\mathcal{S}$ is called the *initial probability vector* of the process.

For i, j in $\mathcal{S}$ and $t \in \mathbb{R}^+$, the transition functions

$$P_{ij}(t) = \mathrm{P}[X(t) = j|X(0) = i]$$

are the solution of the *forward Kolmogorov* equations:

$$\frac{dP(t)}{dt} = P(t)Q,$$

with $P(0) = I$, where the matrix $P(t)$ contains the elements $P_{ij}(t)$ for all $i, j \in \mathcal{S}$ and where the coefficient matrix Q is called the *infinitesimal transition generator* of the process. The interpretation of the entries of the matrix Q is the following.

- For $i \neq j$, Q_{ij} is the instantaneous transition rate from state i to state j. In other words,

  $$Q_{ij}h = \mathrm{P}[X(t+h) = j|X(t) = i] + o(h),$$

  where $o(h)$ has the usual meaning $\lim_{h\to 0} o(h)/h = 0$, thus $Q_{ij}h$ is the probability that the process leaves state i before time $t + h$, starting from state i at time t, and enters j. The entries Q_{ij} are nonnegative and are strictly positive if it is possible to move from i to j in one step.

- The process stays is state i during an interval of time which is exponentially distributed with parameter $q_i = -Q_{ii}$. If $q_i = 0$, then the process stays forever in state i once it has reached this state; i is then called an *absorbing* state. On the diagonal, $Q_{ii} = -\sum_{j \in \mathcal{S}, j \neq i} Q_{ij}$.

Denote by $N_{t,t+h}$ the number of state changes of the Markov process $\{X(t)\}$ in the interval $[t, t+h]$; one has the following characterization:

$$\mathrm{P}[N_{t,t+h} = 0 | X(t) = i] = 1 - q_i h + o(h),$$
$$\mathrm{P}[N_{t,t+h} = 1 | X(t) = i] = q_i h + o(h)$$

and

$$\mathrm{P}[N_{t,t+h} \geq 2 | X(t) = i] = o(h),$$

for some small h and for any i in $\mathcal{S}$. These are easily proved using the fact that the sojourn time in any state i is exponentially distributed with parameter q_i.

We associate an *oriented transition graph* to a Markov process in the following way. The nodes of the graph represent the states of the Markov process, and there is a directed arc from state i to state j, denoted by (i, j), if and only if $Q_{ij} > 0$. A *path* from i to j is a finite sequence of directed arcs $(i, i_1), (i_1, i_2), \ldots, (i_n, j)$.

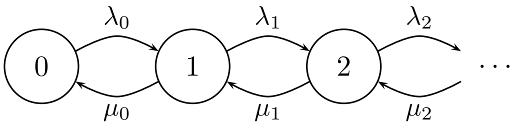
Example A.1.3 *Birth-and-Death process*
We give here an example of a Markov process on the state space $\mathcal{S} = \mathbb{N}$. Consider a population such that, when there are i individuals present in the population:

- a new individual is born after a random interval of time exponentially distributed with parameter λ_i,
- one individual will die after an interval of time which is random and exponentially distributed with parameter μ_i; we set $\mu_0 = 0$.

The infinitesimal transition generator Q of such a Markov process is

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \cdots & \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & \cdots & \\ 0 & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

One may associate with this process the following oriented transition graph:



We say that state l is *accessible* from state k if there is a path that leads from k to l in the transition graph.

Definition A.1.4 *The Markov process $\{X(t) : t \in \mathbb{R}^+\}$ on the state space $\mathcal{S}$ is irreducible if, for all i, j in $\mathcal{S}$, i is accessible from j and j is accessible from i.*

For j in $\mathcal{S}$, we denote by T_j the first passage time to state j, that is,

$$T_j = \inf\{t > 0 : X(t) = j, X(t-) \neq j\}$$

where $X(t-) = \lim_{u \to t,\ u<t} X(u)$. The first passage probability from state i to state j is

$$F_{ij} = \mathrm{P}[T_j < \infty | X(0) = i].$$

Definition A.1.5 *A state j is positive recurrent if and only if $F_{jj} = 1$ and $E[T_j|X(0) = j] < \infty$.*

Corollary A.1.6 *If the Markov process $\{X(t) : t \in \mathbb{R}^+\}$ is irreducible and if one of its states is positive recurrent, then all the states are positive recurrent and $F_{ij} = 1$ for all i, j in $\mathcal{S}$. In that case, the process itself is said to be positive recurrent.*

$\square$

Remark A.1.7 A positive recurrent Markov process is sometimes also called *ergodic*.

The following theorem gives a necessary and sufficient condition for the process to be positive recurrent.

Theorem A.1.8 *Assume that the Markov process* $\{X(t) : t \in \mathbb{R}^+\}$ *is irreducible and non exploding, and denote by* Q *its transition generator. It is positive recurrent if and only if there exists a probability row vector* $\boldsymbol{\pi}$ *such that* $\pi_i > 0$ *for all* i *in* $\mathcal{S}$*, and such that* $\boldsymbol{\pi}$ *solves the system*

$$\begin{cases} \boldsymbol{\pi} Q &= \mathbf{0} \\ \boldsymbol{\pi}\mathbf{1} &= 1 \end{cases}.$$

This vector $\boldsymbol{\pi}$ *is unique and is such that*

$$\pi_j = \lim_{t\to\infty} P[X(t) = j | X(0) = i],$$

for all i *and* j *in* $\mathcal{S}$.

□

The vector $\boldsymbol{\pi}$ is called the *stationary* probability vector of the process. Other terms used are *invariant, steady state, equilibrium* or *asymptotic.*

In case the process is not positive recurrent, we have that

$$\lim_{t\to\infty} \mathrm{P}[X(t) = j | X(0) = i] = 0$$

for all $i, j \in \mathcal{S}$.

A.2 Poisson Processes and the M/M/1 Queue

Consider a continuous-time stochastic process $\{N(t) : t \in \mathbb{R}^+\}$ with state space $\mathcal{S} = \mathbb{N}$. One definition of a *Poisson process* $\{N(t)\}$ *with parameter* λ is that it is a Markov process such that

- $N(0) = 0$,
- the only transitions allowed are from some state i to $i + 1$, with $i \in \mathbb{N}$,
- the sojourn times in each state are exponentially distributed with parameter λ, independently of the state.

The generator of $\{N(t)\}$ is thus given by

$$\begin{bmatrix} -\lambda & \lambda & & \\ & -\lambda & \lambda & \\ & & \ddots & \ddots \\ & & & \end{bmatrix}.$$

An important feature of the Poisson process is that the number of events of the process in the interval $(0,t)$ is Poisson distributed. More specifically, for the process $\{N(t)\}$, we have that

$$\mathrm{P}[N(t)=k]=e^{-\lambda t}\frac{(\lambda t)^k}{k!},$$

for $k\geq 0$, thus the number of events in $(0,t)$ is Poisson distributed with parameter λt.

The Poisson process is widely used to model customer arrivals to a queueing system. The simplest one is the so-called M/M/1 queue which can be described as follows. Customers arrive in the system according to a Poisson process with rate λ. There is one server who serves at rate μ, that is, service times are independent and identically distributed exponential random variables with parameter μ, independent of the arrival process. The generator of the M/M/1 queue is

$$\begin{bmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & \mu & -(\lambda+\mu) & \lambda & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}.$$

Note that it is a special case of the Birth-and-Death process defined in Example A.1.3.

A.3 Censored Markov Processes

Consider now two proper subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$, that is, $\mathcal{S}=\mathcal{A}\cup\mathcal{B}$ and $\mathcal{A}\cap\mathcal{B}=\emptyset$. We partition the generator Q accordingly as

$$Q=\begin{bmatrix} Q_{\mathcal{A}\mathcal{A}} & Q_{\mathcal{A}\mathcal{B}} \\ Q_{\mathcal{B}\mathcal{A}} & Q_{\mathcal{B}\mathcal{B}} \end{bmatrix}.$$

Thus, $Q_{\mathcal{A}\mathcal{A}}$ contains the components Q_{ij} such that $i,j\in\mathcal{A}$, $Q_{\mathcal{A}\mathcal{B}}$ contains the components Q_{ij} such that $i\in\mathcal{A}$ and $j\in\mathcal{B}$, and so forth. We also partition the stationary probability vector $\boldsymbol{\pi}$ according to the decomposition of $\mathcal{S}$, and write that $\boldsymbol{\pi}=(\boldsymbol{\pi}_{\mathcal{A}},\boldsymbol{\pi}_{\mathcal{B}})$ where $\boldsymbol{\pi}_{\mathcal{A}}=(\pi_i:i\in\mathcal{A})$ and $\boldsymbol{\pi}_{\mathcal{B}}=(\pi_i:i\in\mathcal{B})$.

Theorem A.3.1 *Assume that the Markov process $\{X(t):t\in\mathbb{R}^+\}$ is irreducible and let $\boldsymbol{\pi}$ be its stationary probability vector. Let $\mathcal{A}$ and $\mathcal{B}$ be* *two proper sets of $\mathcal{S}$. One has*

$$\boldsymbol{\pi}_{\mathcal{B}} = \boldsymbol{\pi}_{\mathcal{A}} Q_{\mathcal{AB}} N_{\mathcal{B}}$$

where $N_{\mathcal{B}}$ records the expected sojourn time in the states of $\mathcal{B}$ given an initial state in $\mathcal{B}$, before the first visit to $\mathcal{A}$.

□

Definition A.3.2 *The censored process restricted to the set $\mathcal{A}$ is obtained by removing from the original Markov process all the intervals of time during which it is in $\mathcal{B}$; it is denoted by $\{X^{\mathcal{A}}(t) : t \in \mathbb{R}^+\}$.*

The following theorem gives the characterization of the censored process restricted to the set $\mathcal{A}$.

Theorem A.3.3 *Let $\{X(t) : t \in \mathbb{R}^+\}$ be an irreducible and positive recurrent Markov process on the state space $\mathcal{S}$, with generator Q. Let $\mathcal{A}$ and $\mathcal{B}$ be two proper subsets of $\mathcal{S}$.*

The restricted process $\{X^{\mathcal{A}}(t) : t \in \mathbb{R}^+\}$ is an irreducible and positive recurrent Markov process on the states of $\mathcal{A}$. Its generator is given by

$$Q^* = Q_{\mathcal{AA}} + Q_{\mathcal{AB}}(-Q_{\mathcal{BB}})^{-1} Q_{\mathcal{BA}}.$$

Its stationary probability vector is proportional to $\boldsymbol{\pi}_{\mathcal{A}}$, thus

$$\boldsymbol{\pi}_{\mathcal{A}} Q^* = \mathbf{0}.$$

□

A.4 Phase-Type Distributions

Consider a Markov process on the state space $\mathcal{S} = \{0, 1, \ldots, n\}$ with initial probability vector $(\tau_0, \boldsymbol{\tau})$ and infinitesimal generator

$$Q = \begin{bmatrix} 0 & \mathbf{0} \\ \boldsymbol{t} & T \end{bmatrix}$$

where $\boldsymbol{\tau}$ is $1 \times n$, T is $n \times n$ and $\boldsymbol{t}$ is $n \times 1$. The state 0 is absorbant, while all the other states are transient and lead to 0.

Definition A.4.1 *The distribution of the time X until absorption into the absorbing state 0 is called a phase-type distribution with representation $(\boldsymbol{\tau}, T)$. We say that X is PH$(\boldsymbol{\tau}, T)$.*

The following theorem gives the distribution and density functions of a PH random variable, as well as its moments.

Theorem A.4.2 *Assume that X is PH$(\boldsymbol{\tau}, T)$. Its distribution function is given by*

$$F(x) = 1 - \boldsymbol{\tau} e^{Tx}\mathbf{1}, \quad \textit{for } x \geq 0,$$

and its density function is given by

$$f(x) = \boldsymbol{\tau} e^{Tx}\boldsymbol{t}, \quad \textit{for } x > 0.$$

Its moments are given by

$$E[X^k] = k!\boldsymbol{\tau}(-T^{-1})^k\mathbf{1}, \quad \textit{for } k \geq 1.$$

$\square$

Remark A.4.3 The matrix exponential is defined as $e^M = \sum_{n\geq 0} \frac{M^n}{n!}$.